UDC 519.62; 004.032.26
IRSTI 27.41.19; 28.23.15
DOI: https://doi.org/10.66941/3106-9037/2026/iss3pp7-16

Kim A.E.
International Information Technology University, Almaty, Kazakhstan

PARAMETRIZATION METHOD AND PHYSICS-INFORMED NEURAL NETWORKS FOR THE DUFFING EQUATION

Kim Andrey Edikovich, Master of Technical Sciences, Assistant Professor, Department of Mathematical and Computer Modeling, International Information Technology University, 34/1 Manas St., Almaty 050040, Kazakhstan.
E-mail: kandreym@icloud.com; https://doi.org/: 0009-0006-3369-7694

Abstract. *Introduction.* Physics-informed neural networks (PINNs) are increasingly proposed as universal solvers of differential equations, but their reliability for nonlinear oscillators with several periodic solutions is insufficiently studied. *Aims and objectives.* To compare D.S. Dzhumabaev's parametrization method with PINNs for periodic solutions of the Duffing equation in terms of accuracy, cost, robustness and control over coexisting solutions. *Results.* The parametrization method reached errors of 10^{-8}-10^{-7} in hundredths of a second; eight subintervals raised the share of convergent initial approximations under strong softening nonlinearity from 0-9% to 61-100%; with arclength continuation and Floquet multipliers, the method reproduced the full hysteresis of the frequency response. PINNs reached errors of 10^{-6}-10^{-2} in 20-40 s and in some regimes converged, with an equally small loss, to a different periodic solution. *Conclusion.* PINN solutions require verification, which the parametrization method provides.

Keywords: Duffing equation, periodic boundary value problem, parametrization method, physics-informed neural networks, Newton's method, frequency response.

А.Э. Ким
Международный университет информационных технологий, Алматы, Казахстан

МЕТОД ПАРАМЕТРИЗАЦИИ И ФИЗИЧЕСКИ ИНФОРМИРОВАННЫЕ НЕЙРОННЫЕ СЕТИ ДЛЯ УРАВНЕНИЯ ДУФФИНГА

Ким Андрей Эдикович, магистр технических наук, ассистент-профессор кафедры математического и компьютерного моделирования, Международный университет информационных технологий, ул. Манаса, 34/1, Алматы 050040, Казахстан.
E-mail: kandreym@icloud.com; https://doi.org/: 0009-0006-3369-7694

Аннотация. *Вступление.* Физически информированные нейронные сети (PINN) все чаще предлагаются как универсальные решатели дифференциальных уравнений, однако их надежность для нелинейных осцилляторов с несколькими периодическими решениями изучена недостаточно. *Цели и задачи.* Сравнить метод параметризации Д.С. Джумабаева с PINN при нахождении периодических решений уравнения Дуффинга по точности, затратам, устойчивости и контролю над сосуществующими решениями. *Результаты.* Метод параметризации достиг погрешности 10^{-8}-10^{-7} за сотые доли секунды; разбиение на восемь частей при сильной мягкой нелинейности повысило долю сходящихся начальных приближений с 0-9% до 61-100%; в сочетании с продолжением по длине дуги и мультипликаторами Флоке метод воспроизвел полный гистерезис амплитудно-частотной характеристики. PINN достигли погрешности 10^{-6}-10^{-2} за 20-40 с и в ряде режимов сходились к другому периодическому решению при столь же малой функции потерь. *Заключение.* Решения PINN нуждаются в верификации, которую обеспечивает метод параметризации.

Ключевые слова: уравнение Дуффинга, периодическая краевая задача, метод параметризации, физически информированные нейронные сети, метод Ньютона, амплитудно-частотная характеристика.

А.Э. Ким
Халықаралық ақпараттық технологиялар университеті, Алматы, Қазақстан

ДУФФИНГ ТЕҢДЕУІ ҮШІН ПАРАМЕТРЛЕУ ӘДІСІ ЖӘНЕ ФИЗИКАЛЫҚ АҚПАРАТТАНДЫРЫЛҒАН НЕЙРОНДЫҚ ЖЕЛІЛЕР

Ким Андрей Эдикович, техника ғылымдарының магистрі, Халықаралық ақпараттық технологиялар университетінің математикалық және компьютерлік модельдеу кафедрасының ассистент-профессоры, Манас к-сі, 34/1, Алматы 050040, Қазақстан.
E-mail: kandreym@icloud.com; https://doi.org/: 0009-0006-3369-7694

Аннотация. *Кіріспе.* Физикалық ақпараттандырылған нейрондық желілер (PINN) дифференциалдық теңдеулерді шешудің әмбебап құралы ретінде жиі ұсынылады, алайда бірнеше периодты шешімі бар сызықтық емес осцилляторлар үшін олардың сенімділігі жеткілікті зерттелмеген. *Мақсаты мен міндеттері.* Дуффинг теңдеуінің периодты шешімдерін табуда Д.С. Жұмабаевтың параметрлеу әдісін PINN-мен дәлдік, есептеу шығын, орнықтылық және қатар өмір сүретін шешімдерді бақылау тұрғысынан салыстыру. *Нәтижелер.* Параметрлеу әдісі секундтың жүзден бір бөлігінде 10^{-8}-10^{-7} қателікке жетті; күшті жұмсарушы сызықты емес жағдайда сегіз бөлікке бөлу жинақталатын бастапқы жуықтаулар үлесін 0-9%-дан 61-100%-ға дейін арттырды; доға ұзындығы бойынша жалғастыру және Флоке мультипликаторларымен бірге әдіс амплитудалық-жиіліктік сипаттаманың толық гистерезисін қалпына келтірілді. PINN 20-40 секундта 10^{-6}-10^{-2} қателікке жетті және кейбір режимдерде шығын функциясы дәл сондай кіші болғанымен басқа периодты шешімге жинақталды. *Қорытынды.* PINN шешімдері тексеруді қажет етеді, оны параметрлеу әдісі қамтамасыз етеді.

Түйін сөздер: дуффинг теңдеуі, периодты шеттік есеп, параметрлеу әдісі, физикалық ақпараттандырылған нейрондық желілер, Ньютон әдісі, амплитудалық-жиіліктік сипаттама.

For citation: Kim A.E. (2026). Parametrization method and physics-informed neural networks for the Duffing equation. *Science of Intelligent Systems*, [Vol.(No.)], [pages]. (In Eng.). https://doi.org/[to be assigned]

Introduction

The Duffing equation, introduced to describe forced oscillations with a nonlinear restoring force (Duffing, 1918), is a standard model of nonlinear vibrations of beams, pendulums and electrical circuits. It exhibits jump resonance, coexistence of several periodic regimes and chaos (Holmes, 1979; Ueda, 1979; Guckenheimer, Holmes, 1983; Kovacic, Brennan, 2011). Steady-state vibrations correspond to periodic solutions, and computing them requires solving a nonlinear boundary value problem in which the state at the end of the period coincides with the state at its beginning. Such a problem may have several solutions, and its numerical solution is sensitive to the initial approximation (Keller, 1968; Ascher et al., 1995).

Numerical methods for boundary value problems for ordinary differential equations include shooting, multiple shooting, finite-difference and collocation methods (Keller, 1968; Ascher et al., 1995; Stoer, Bulirsch, 2002). Shooting is simple but sensitive to the growth of solutions of the Cauchy problem; multiple shooting limits this growth by splitting the interval; adaptive collocation controls the residual and refines the mesh automatically (Kierzenka, Shampine, 2001). For periodic regimes of oscillators, path-following methods trace branches of solutions and detect folds and bifurcations (Keller, 1977; Allgower, Georg, 2003), while harmonic balance provides analytical approximations of the frequency response (Nayfeh, Mook, 1979; Krack, Gross, 2019).

The parametrization method of Dzhumabaev (1989) was created to obtain criteria of unique solvability of linear boundary value problems in terms of the initial data. Dzhumabaev and Temesheva (2007) extended it to nonlinear problems: they proved a sharpened local Hadamard theorem, proposed a family of algorithms with convergence conditions in terms of the initial data and established a criterion for an isolated solution; these results were refined later (Dzhumabaev, Temesheva, 2013; 2018), and modified algorithms and a numerical method were proposed (Temesheva, 2020; Temesheva et al.,

2021). An essential feature of this theory is that its convergence conditions improve as the partition is refined. What has been missing is a computational picture of this effect for a concrete oscillator and a comparison with modern approaches.

Physics-informed neural networks go back to the use of neural networks for solving differential equations (Lagaris et al., 1998) and were popularised by Raissi et al. (2019); their place in scientific machine learning is reviewed by Karniadakis et al. (2021). Known difficulties include imbalanced gradients between loss terms (Wang et al., 2021) and failure to learn solutions in more demanding regimes (Krishnapriyan et al., 2021). Exact enforcement of periodicity through Fourier-type inputs removes one source of error (Dong, Ni, 2021). However, the loss of a PINN measures only how well the equation is satisfied, not which solution is approximated, and for problems with several solutions this question is rarely addressed.

The relevance of the study follows from this situation. Intelligent systems increasingly embed neural solvers of differential equations, and the question of when such solvers can be trusted is practical. Nonlinear oscillators with coexisting periodic regimes are a demanding test because a solver may return a correct-looking but physically different solution. To the author's knowledge, a direct comparison of the parametrization method with PINNs has not been reported, and quantitative data on how the partition in the parametrization method affects the practical domain of convergence of Newton's method for the Duffing oscillator are scarce.

The object of the research is the periodic boundary value problem for the Duffing equation. The subject is the accuracy, cost, robustness and control over the multiplicity of solutions of the parametrization method and of PINNs. The aim is to compare the two approaches quantitatively. The tasks are: (1) to implement the parametrization method with Runge-Kutta integration, Newton's method, continuation and Floquet stability analysis; (2) to verify it on a problem with a known exact solution; (3) to measure the domain of convergence as a function of the number of subintervals and to relate it to the conditions of Newton-Kantorovich-type theorems; (4) to compute the frequency response of the physical damped Duffing oscillator; (5) to train PINNs on the same problems and compare the results. The hypothesis is that PINNs approach the accuracy of classical methods only at much higher cost and do not control which of the coexisting periodic solutions they find, whereas the parametrization method with a suitable partition and continuation provides both accuracy and control.

The scientific novelty consists in the first, to the author's knowledge, systematic comparison of the Dzhumabaev parametrization method with physics-informed neural networks; in the quantification, with confidence intervals, of the effect of the partition on the domain of convergence together with the Lipschitz constant of the Jacobian of the parameterized system; and in the use of the matrices computed by the method for Floquet stability analysis and arclength continuation of the frequency response.

Materials and methods

Research questions. RQ1: how accurate and how expensive is each approach? RQ2: how does the number of subintervals affect the domain of convergence of the parametrization method, and how does this relate to theoretical convergence conditions? RQ3: can each approach reproduce the coexisting periodic solutions of the physical Duffing oscillator? Hypothesis: as stated in the Introduction, PINNs are less accurate and more expensive and do not control the selection among coexisting solutions, whereas the parametrization method provides accuracy and, combined with continuation, control.

Problem 1 (verification problem). Written as a first-order system, the forced Duffing equation on [0, T] is

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = -\delta x_2 - \sigma^2 x_1 - \varepsilon\gamma x_1^3 + B\cos\omega t + g(t), \qquad t \in [0, T], \tag{1}$$

with the periodic boundary condition

$$x(0) = x(T), \qquad x = (x_1, x_2)^{\mathrm{T}}, \tag{2}$$

where δ is the damping coefficient, σ the natural frequency, ε the nonlinearity coefficient, γ = +1 for hardening and γ = −1 for softening restoring force, and B cos ωt the external forcing. In Problem 1, δ = 0, T = 1, σ = 1, B = 2, ω = 2π, and ε ∈ [0.1, 10]. Following the method of manufactured solutions (Roache, 2002), the source term

$$g(t) = (\sigma^2 - 4\pi^2)\sin 2\pi t + \varepsilon\gamma \sin^3 2\pi t - B\cos 2\pi t \tag{3}$$

makes x*(t) = (sin 2πt, 2π cos 2πt) an exact solution, so that errors can be measured directly.

Problem 2 (physical problem). The damped forced Duffing oscillator without any artificial source term,

$$\ddot{x} + \delta\dot{x} + \sigma^2 x + \varepsilon\gamma x^3 = B\cos\omega t, \qquad x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T), \quad T = \frac{2\pi}{\omega}, \tag{4}$$

with δ = 0.1, σ = 1, εγ = 1 and B = 0.3, was studied for forcing frequencies ω ∈ [0.6, 3.2]. Its periodic solutions form the frequency response curve, which for a hardening spring contains a region of coexisting solutions (Nayfeh, Mook, 1979).

Parametrization method. The interval is divided as $0 = t_0 < t_1 < \ldots < t_N = T$, the parameters $\lambda_r = x(t_{r-1})$ and the functions $u_r(t) = x(t) - \lambda_r$ are introduced, and the problem becomes equivalent to the problem with parameters (Dzhumabaev, 1989; Dzhumabaev, Temesheva, 2007)

$$\frac{du_r}{dt} = f(t, u_r + \lambda_r), \qquad u_r(t_{r-1}) = 0, \qquad r = \overline{1,N}, \tag{5}$$

$$\lambda_1 - \lambda_N - u_N(T) = 0, \qquad \lambda_r + u_r(t_r) - \lambda_{r+1} = 0, \quad r = \overline{1,N-1}, \tag{6}$$

where f is the right-hand side of (1). The boundary condition and the matching conditions (6) form a system Q(λ) = 0 of 2N equations, solved by Newton's method (Kantorovich, Akilov, 1982; Deuflhard, 2011)

$$\frac{\partial Q}{\partial \lambda}\left(\lambda^{(k)}\right)\Delta\lambda^{(k)} = -Q\left(\lambda^{(k)}\right), \qquad \lambda^{(k+1)} = \lambda^{(k)} + \Delta\lambda^{(k)}. \tag{7}$$

The blocks of the Jacobian are obtained from the variational equations

$$\frac{dZ_r}{dt} = f_x\left(t, \lambda_r + u_r(t)\right)Z_r, \quad Z_r(t_{r-1}) = I, \qquad f_x = \begin{pmatrix} 0 & 1 \\ -\sigma^2 - 3\varepsilon\gamma x_1^2 & -\delta \end{pmatrix}, \tag{8}$$

since $\partial(\lambda_r + u_r(t_r))/\partial\lambda_r = Z_r(t_r)$. Problems (5) and (8) were integrated jointly by the classical fourth-order Runge-Kutta method (Hairer et al., 1993; Butcher, 2016); iterations stopped when $\max|\Delta\lambda^{(k)}| < 10^{-10}$. For N = 1 the algebraic system coincides with that of single shooting and for N > 1 it is algebraically related to multiple shooting (Stoer, Bulirsch, 2002). The parametrization method contributes the theory of the problem with parameters — solvability criteria, conditions of convergence and of the existence of an isolated solution (Dzhumabaev, Temesheva, 2007; 2013; 2018) — which is used below to interpret the role of the partition.

Stability and continuation. The monodromy matrix of a periodic solution is the product of the matrices already computed by the method,

$$M = Z_N(t_N)\, Z_{N-1}(t_{N-1}) \cdots Z_1(t_1), \tag{9}$$

and the solution is asymptotically stable if all Floquet multipliers, the eigenvalues of M, lie inside the unit circle (Nayfeh, Balachandran, 1995). The frequency response of Problem 2 was traced by pseudo-arclength continuation in the variables y = (λ, ω) (Keller, 1977; Allgower, Georg, 2003), which adds to Q(λ, ω) = 0 the condition

$$\dot{y}_j^{\mathrm{T}}\left(y_{j+1} - y_j\right) - \Delta s = 0, \tag{10}$$

where $\dot{y}_j$ is the unit tangent and Δs the step; this allows the curve to pass around fold points where natural continuation in ω fails. Natural sweeps upward and downward in ω with step 0.02 were also performed. In Problem 1, natural continuation in ε from the linear problem (ε = 0) in ten steps was used as a baseline for choosing initial approximations.

Convergence conditions. Newton's method converges from any $\lambda^{(0)}$ with $|\lambda^{(0)} - \lambda| < 2/(3\beta K)$, *where β is the norm of the inverse Jacobian at the solution and K the Lipschitz constant of the Jacobian (Traub, Woźniakowski, 1979). To relate this condition to the partition, β and K were estimated at the solution of Problem 1 for N = 1, 2, 4, 8, with K computed as the largest ratio* $|\partial Q/\partial\lambda(\lambda) - \partial Q/\partial\lambda(\lambda)| \,/\, |\lambda - \lambda|$ *over 120 random points with* $|\lambda - \lambda| \le 1$ in the maximum norm.

Physics-informed neural networks. The solution x_1 was approximated by a fully connected network with two hidden layers of 32 neurons and tanh activation. In the variant with exact periodicity the network inputs were sin ωt and cos ωt, so that every network output satisfies the periodic condition (Dong, Ni, 2021); in the variant with a soft constraint the input was t and the periodicity mismatch was added to the loss with weight 10. The loss was the mean squared residual of the equation at 256 collocation points,

$$\mathcal{L}(\theta) = \frac{1}{n}\sum_{k=1}^{n}(\hat{x}_\theta''(t_k) + \delta\hat{x}_\theta'(t_k) + \sigma^2\hat{x}_\theta(t_k) + \varepsilon\gamma\hat{x}_\theta^3(t_k) - B\cos\omega t_k - g(t_k))^2, \quad (11)$$

with derivatives computed by automatic differentiation (Maclaurin et al., 2015). Training used 2000 iterations of Adam (Kingma, Ba, 2015) followed by up to 3000 iterations of L-BFGS (Liu, Nocedal, 1989), as recommended for PINNs (Raissi et al., 2019). Each configuration was trained from three (Problem 1) or five (Problem 2) random initialisations.

Experiments and material. E1 (accuracy): errors of the parametrization method for Runge-Kutta steps h = 0.1-0.00625 and N = 1, 4. E2 (robustness): for $\gamma = -1$, $\varepsilon \in \{1, 5, 10\}$ and $\gamma = 1$, $\varepsilon \in \{5, 10\}$, Newton's method was started from 200 random constant initial approximations $x^{(0)} \equiv c$ with c uniform in $[-A, A]^2$, $A \in \{2, 6\}$; each run was classified as convergence to x* (error below 10^{-5}), convergence to another periodic solution, or divergence, and 95% Wilson confidence intervals were computed (Wilson, 1927). The same starts were given to the collocation solver solve_bvp (Kierzenka, Shampine, 2001; Virtanen et al., 2020). E3 (convergence conditions): β and K as described above. E4 (physical problem): frequency response of Problem 2 with N = 4 and 400 Runge-Kutta steps per period, compared with the first-harmonic balance approximation (Nayfeh, Mook, 1979). E5 (PINNs): six regimes of Problem 1 ($\varepsilon \in \{0.1, 1, 10\}$, $\gamma = \pm 1$) and Problem 2 at $\omega = 1.0$ and $\omega = 1.5$. Computations were performed in Python 3 with NumPy and SciPy on a single CPU core. The code used for all computations is available from the author upon request.

Results and discussion

Accuracy, convergence rate and cost of the parametrization method

Table 1 shows the errors for Problem 1 with $\varepsilon = 0.1$, $\gamma = 1$. Halving the Runge-Kutta step reduced the error about sixteen times, and the observed order tended to 4.00, confirming the theoretical order of the scheme. The partition N = 4 gave the same error as N = 1 for $h \le 0.05$.

Step h	Error, N = 1	Error, N = 4	Ratio E(2h)/E(h)	Observed order
0.1	$3.71\cdot10^{-4}$	$9.15\cdot10^{-4}$	-	-
0.05	$2.29\cdot10^{-5}$	$2.29\cdot10^{-5}$	16.2	4.02
0.025	$1.42\cdot10^{-6}$	$1.42\cdot10^{-6}$	16.1	4.01
0.0125	$8.89\cdot10^{-8}$	$8.89\cdot10^{-8}$	16.0	4.00
0.00625	$5.56\cdot10^{-9}$	$5.56\cdot10^{-9}$	16.0	4.00

Table 1 - Maximum error of the parametrization method for Problem 1 ($\varepsilon = 0.1$, $\gamma = 1$)

Newton's method showed quadratic convergence near the solution. For $\varepsilon = 1$, $\gamma = -1$, N = 4 and a distant initial approximation, the increments of the last iterations were $1.2\cdot10^{-2}$, $1.2\cdot10^{-4}$, $9.4\cdot10^{-9}$ and $2.1\cdot10^{-15}$: each was approximately the square of the previous one. One solution of Problem 1 with N = 4 and h = 0.01 took 0.023 s, comparable to 0.019 s for the collocation solver.

Domain of convergence and the partition

Table 2 and Figure 1 present the results of experiment E2 for softening nonlinearity. Without partitioning, Newton's method converged from 0-9% of the initial approximations; with eight subintervals the share rose to 61-100%. For $\varepsilon = 10$ and A = 2 it increased from 0% (N = 1 and N = 2) to 70% (N = 4) and 95.5% (N = 8). The confidence intervals of N = 1 and N = 8 do not overlap in any configuration. The apparent non-monotonicity at $\varepsilon = 5$, A = 2 (100% for N = 4 and 97% for N = 8) lies within the confidence intervals; it reflects the larger number of unknown parameters that must be approximated simultaneously when N grows.

ε	A	N = 1	N = 2	N = 4	N = 8	Collocation
1	2	4 [2-8]	73.5 [67-79]	94 [90-97]	98.5 [96-99]	100 [98-100]
1	6	9 [6-14]	25 [20-31]	94.5 [90-97]	100 [98-100]	100 [98-100]
5	2	0.5 [0-3]	11 [7-16]	100 [98-100]	97 [94-99]	100 [98-100]

ε	A	N = 1	N = 2	N = 4	N = 8	Collocation
5	6	0.5 [0-3]	3.5 [2-7]	38 [32-45]	88 [83-92]	100 [98-100]
10	2	0 [0-2]	0 [0-2]	70 [63-76]	95.5 [92-98]	100 [98-100]
10	6	0 [0-2]	0 [0-2]	19 [14-25]	61 [54-67]	100 [98-100]

Table 2 - Softening nonlinearity (γ = −1): share of 200 initial approximations converging to the solution, % [95% confidence interval]

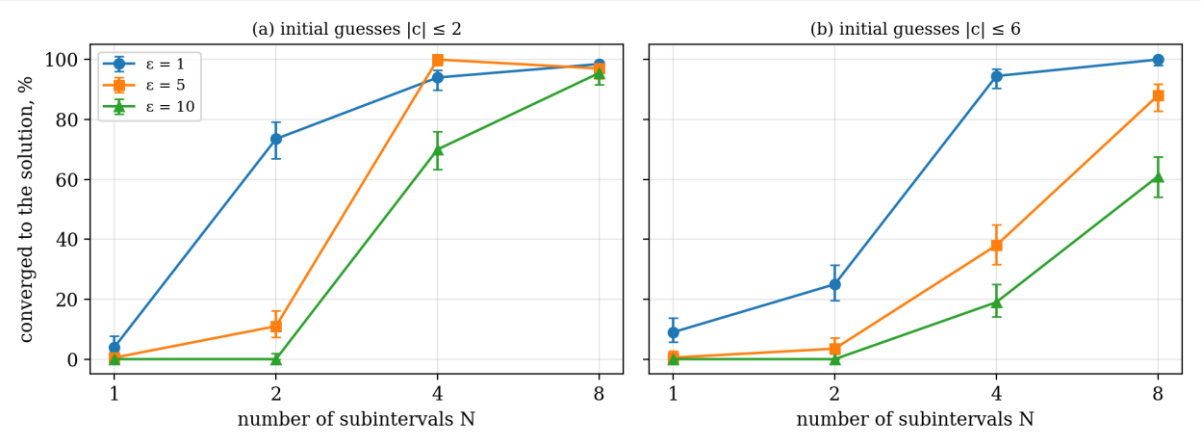


Figure 1 - Share of initial approximations converging to the solution as a function of the number of subintervals N, with 95% confidence intervals: (a) |c| ≤ 2, (b) |c| ≤ 6

The cause of this effect is visible in the convergence conditions. For ε = 10, γ = −1, the Lipschitz constant K of the Jacobian over the ball $|\lambda - \lambda^*| \le 1$ was unbounded for N = 1 and N = 2, because for some points of the ball the Cauchy problems blew up before the end of the period, whereas it was finite for N = 4 (K = 407) and N = 8 (K = 20.8). For ε = 5 the values were unbounded, $1.5 \cdot 10^7$, 32.2 and 7.6 for N = 1, 2, 4, 8. For weak hardening nonlinearity (ε = 0.1) K was small for all N (0.60, 0.48, 0.20 and 0.11). The norm of the inverse Jacobian β grew with N (2.2, 2.7, 5.5 and 10.0 for ε = 10). Partitioning therefore makes the system Q(λ) = 0 well defined and smooth on a large neighbourhood of the solution, which is exactly what theorems of the Newton-Kantorovich type require, at the cost of a moderate growth of β. The guaranteed radius $2/(3\beta K)$ is conservative ($3.2 \cdot 10^{-3}$ for ε = 10, N = 8), but it is non-negligible only for N ≥ 4, in agreement with the experiment.

Two further observations limit the role of the partition. First, natural continuation in ε from the linear problem converged to the solution in all five strongly nonlinear regimes both for N = 1 and for N = 4, with 22 Newton iterations in total and errors of $2 \cdot 10^{-8}$-$5 \cdot 10^{-7}$. The partition is therefore decisive when no continuous path from a known solution is available, for example when coexisting solutions are sought. Second, for hardening nonlinearity (ε = 5 and ε = 10) the partition N = 4 raised convergence to the solution from 1% and 0% to 60.5% and 40.5%, but N = 8 gave no further gain, and many runs converged to other periodic solutions; collocation reached 75.5% and 48.5%, with 21.5% and 43% of runs ending at other solutions. Collocation was the most robust method under softening nonlinearity (100%).

Physical Duffing oscillator: frequency response and stability

For Problem 2 the parametrization method combined with continuation reproduced the classical picture of a hardening Duffing oscillator (Figure 2). In the upward sweep the amplitude followed the upper branch and dropped from 1.67 to 0.15 between ω = 1.72 and 1.74; in the downward sweep it jumped from 0.46 to 1.15 between ω = 1.34 and 1.32. Arclength continuation passed around both folds, located at ω ≈ 1.780 (amplitude 1.723) and ω ≈ 1.324 (amplitude 0.597), and traced the middle branch: 78 of 208 computed points had a Floquet multiplier outside the unit circle and were unstable, while all points of the upper and lower branches were stable. The natural sweeps left the upper branch slightly before the fold, because a step of 0.02 near the fold carried Newton's method to the lower branch, whereas arclength continuation located the fold precisely; at most 13 Newton iterations per step were needed. The amplitudes agreed with the harmonic balance approximation within 1.5% (at ω = 1.5:

0.247, 1.19 and 1.391 against 0.247, 1.180 and 1.371), the difference being due to higher harmonics neglected by the approximation.

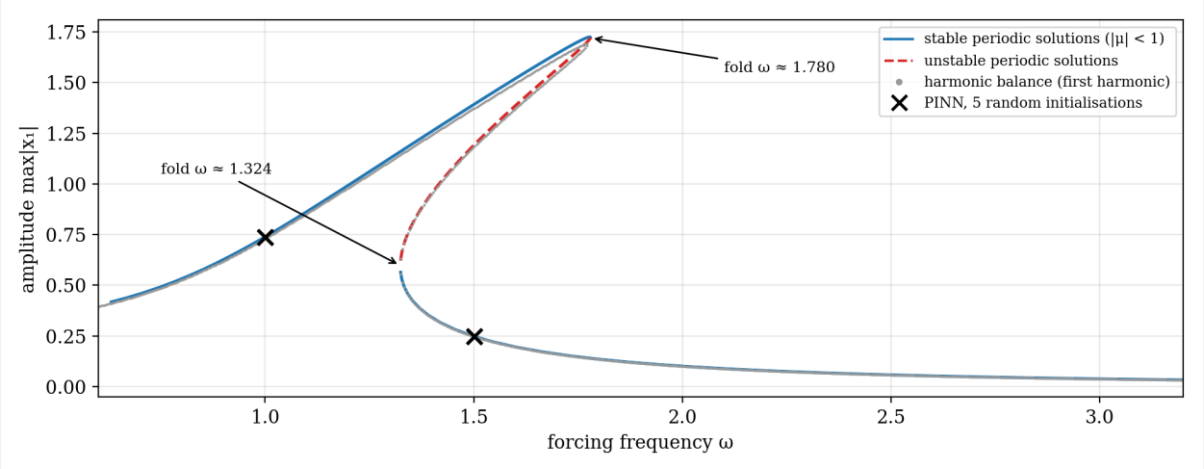


Figure 2 - Frequency response of the damped Duffing oscillator computed by the parametrization method with arclength continuation, compared with harmonic balance and with PINN solutions

Comparison with physics-informed neural networks

Table 3 compares the two approaches on Problem 1. With exactly periodic inputs, PINNs reached errors of $5\cdot10^{-6}$-$6\cdot10^{-4}$ in about 22 s of training; with the soft periodicity constraint the errors were 10^{-4}-$9\cdot10^{-3}$ after about 40 s. The parametrization method reached errors of $2\cdot10^{-8}$-$5\cdot10^{-7}$ in hundredths of a second, that is, it was two to five orders of magnitude more accurate and roughly 1,000-1,700 times faster.

ε	γ	Parametrization method	PINN, exact periodicity	PINN, soft constraint
0.1	+1	$3.6\cdot10^{-8}$	$3.6\cdot10^{-5}$ [$5.2\cdot10^{-6}$-$8.8\cdot10^{-5}$]	$9.6\cdot10^{-4}$ [$7.1\cdot10^{-4}$-$1.0\cdot10^{-3}$]
0.1	−1	$3.3\cdot10^{-8}$	$7.3\cdot10^{-5}$ [$2.6\cdot10^{-5}$-$1.9\cdot10^{-4}$]	other solution (3 of 3)
1	+1	$5.8\cdot10^{-8}$	$6.5\cdot10^{-6}$ [$4.5\cdot10^{-6}$-$1.5\cdot10^{-5}$]	$4.3\cdot10^{-4}$ [$1.5\cdot10^{-4}$-$5.9\cdot10^{-4}$]
1	−1	$2.3\cdot10^{-8}$	$7.8\cdot10^{-5}$ [$4.5\cdot10^{-5}$-$5.6\cdot10^{-4}$]	$5.8\cdot10^{-3}$ [$4.6\cdot10^{-3}$-$8.9\cdot10^{-3}$]
10	+1	$3.4\cdot10^{-7}$	other solution (3 of 3)	$1.6\cdot10^{-4}$ [$1.0\cdot10^{-4}$-$2.7\cdot10^{-4}$]
10	−1	$4.7\cdot10^{-7}$	$8.7\cdot10^{-6}$ [$7.7\cdot10^{-6}$-$1.1\cdot10^{-5}$]	$2.3\cdot10^{-4}$ [$2.3\cdot10^{-4}$-$3.0\cdot10^{-4}$]

Table 3 - Maximum error for Problem 1: Parametrization method (N = 4, h = 0.01) and PINNs (median [minimum-maximum] over three initialisations)

The most important finding concerns reliability rather than accuracy. In two of the twelve PINN configurations, all three training runs converged to a different periodic solution: the error with respect to x* was 0.59 and 3.0, while the final loss (10^{-9} and 10^{-7}) was as small as in the successful runs. A small loss therefore does not show that the intended solution has been found. In Problem 2 the same effect appeared in physical form. At ω = 1.0, where the periodic solution is unique, all five PINN runs gave the amplitude 0.73830, which coincides with the value 0.73828 of the parametrization method to within $3\cdot10^{-5}$. At ω = 1.5, where three periodic solutions coexist, all five runs converged to the lower branch (0.2474), and none found the upper stable branch (1.391), which is the response actually observed after an upward frequency sweep, or the unstable branch. The parametrization method found all three solutions and determined their stability.

Comparison with earlier studies. The failure of PINNs to select the intended solution complements the known failure modes of PINNs, which have so far been attributed mainly to optimisation difficulties and imbalanced gradients (Wang et al., 2021; Krishnapriyan et al., 2021): here the optimisation succeeded, yet the network converged to another solution of the same problem. The computed frequency response agrees with the classical analytical description of the hardening Duffing oscillator

(Nayfeh, Mook, 1979; Kovacic, Brennan, 2011), and the gain from partitioning agrees with the theory of the parametrization method, in which the convergence conditions improve as the partition is refined (Dzhumabaev, Temesheva, 2007). In this sense the study confirms the theoretical role of the partition computationally and transfers it to a new task — the verification of neural solvers.

These results support the hypothesis and suggest a division of roles. PINNs are attractive when a mesh-free, differentiable approximation is needed or when the solver is embedded in a larger learning system. Their output, however, must be verified, and the parametrization method provides the necessary tools: an accurate reference solution, the residual of the matching conditions, Floquet multipliers that classify stability and continuation that reveals coexisting solutions. A hybrid intelligent system can use the parametrization method to verify a PINN solution or to supply it with a target branch.

Conclusion

The aim of the study was achieved by implementing the Dzhumabaev parametrization method with Runge-Kutta integration, Newton's method, continuation and Floquet analysis, and by comparing it with physics-informed neural networks on two Duffing problems.

The main results are as follows. The parametrization method reached errors of 10^{-8}-10^{-7} with fourth-order convergence in the step and quadratic convergence of Newton's method, at a cost of hundredths of a second. Under strong softening nonlinearity, eight subintervals raised the share of convergent initial approximations from 0-9% to 61-100%; this effect is explained by the Lipschitz constant of the Jacobian, which is unbounded without partitioning and becomes finite for $N \geq 4$. Combined with arclength continuation and Floquet multipliers, the method reproduced the full frequency response of the damped Duffing oscillator with both folds and the unstable branch. PINNs were two to five orders of magnitude less accurate and roughly a thousand times slower, and in some regimes converged, with an equally small loss, to a periodic solution different from the intended one; in the region of coexisting solutions they found only one of three.

The study shows that neural solvers of differential equations should be accompanied by classical verification when solutions are not unique, and that the parametrization method is well suited for this role. The results can be applied in teaching numerical methods and in the design of intelligent systems that use neural solvers for nonlinear oscillators. Further work includes training PINNs on selected branches using information from the parametrization method, adaptive choice of the partition from the growth of the variational matrices, and extension to systems with several degrees of freedom.

Author contributions

Kim A.E. formulated the problem, implemented the parametrization method and the physics-informed neural networks, performed the computations, analysed the results and wrote the manuscript.

Funding information

This research received no external funding. The author thanks D.S. Dzhumabaev for scientific guidance.

References

1. Allgower E.L., Georg K. (2003). *Introduction to Numerical Continuation Methods*. Philadelphia: SIAM. 388 p.
2. Ascher U.M., Mattheij R.M.M., Russell R.D. (1995). *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*. Philadelphia: SIAM. 595 p.
3. Butcher J.C. (2016). *Numerical Methods for Ordinary Differential Equations*. 3rd ed. Chichester: Wiley. 513 p.
4. Deuflhard P. (2011). *Newton Methods for Nonlinear Problems: Affine Invariance and Adaptive Algorithms*. Berlin: Springer. 424 p. https://doi.org/10.1007/978-3-642-23899-4
5. Dong S., Ni N. (2021). A method for representing periodic functions and enforcing exactly periodic boundary conditions with deep neural networks. *Journal of Computational Physics*, 435, 110242. https://doi.org/10.1016/j.jcp.2021.110242
6. Duffing G. (1918). *Erzwungene Schwingungen bei veränderlicher Eigenfrequenz und ihre technische Bedeutung* [Forced oscillations with variable natural frequency and their technical significance]. Braunschweig: Vieweg. 134 p. (In German)

7. Dzhumabaev D.S. (1989). Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation. *USSR Computational Mathematics and Mathematical Physics*, 29(1), 34-46.
8. Dzhumabaev D.S., Temesheva S.M. (2007). A parametrization method for solving nonlinear two-point boundary value problems. *Computational Mathematics and Mathematical Physics*, 47(1), 37-61.
9. Dzhumabaev D.S., Temesheva S.M. (2013). Necessary and sufficient conditions for the existence of an "isolated" solution of a nonlinear two-point boundary-value problem. *Journal of Mathematical Sciences*, 194(4), 341-353.
10. Dzhumabaev D.S., Temesheva S.M. (2018). Criteria for the existence of an isolated solution of a nonlinear boundary-value problem. *Ukrainian Mathematical Journal*, 70(3), 410-421.
11. Guckenheimer J., Holmes P. (1983). *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. New York: Springer. 459 p.
12. Hairer E., Nørsett S.P., Wanner G. (1993). *Solving Ordinary Differential Equations I: Nonstiff Problems*. 2nd ed. Berlin: Springer. 528 p. https://doi.org/10.1007/978-3-540-78862-1
13. Holmes P. (1979). A nonlinear oscillator with a strange attractor. *Philosophical Transactions of the Royal Society of London. Series A*, 292(1394), 419-448. https://doi.org/10.1098/rsta.1979.0068
14. Kantorovich L.V., Akilov G.P. (1982). *Functional Analysis*. 2nd ed. Oxford: Pergamon Press. 589 p.
15. Karniadakis G.E., Kevrekidis I.G., Lu L., Perdikaris P., Wang S., Yang L. (2021). Physics-informed machine learning. *Nature Reviews Physics*, 3(6), 422-440. https://doi.org/10.1038/s42254-021-00314-5
16. Keller H.B. (1968). *Numerical Methods for Two-Point Boundary-Value Problems*. Waltham: Blaisdell. 184 p.
17. Keller H.B. (1977). Numerical solution of bifurcation and nonlinear eigenvalue problems. In: P.H. Rabinowitz (Ed.), *Applications of Bifurcation Theory* (pp. 359-384). New York: Academic Press.
18. Kierzenka J., Shampine L.F. (2001). A BVP solver based on residual control and the MATLAB PSE. *ACM Transactions on Mathematical Software*, 27(3), 299-316. https://doi.org/10.1145/502800.502801
19. Kingma D.P., Ba J. (2015). Adam: a method for stochastic optimization. In: *Proceedings of the 3rd International Conference on Learning Representations (ICLR)*. San Diego. https://arxiv.org/abs/1412.6980
20. Kovacic I., Brennan M.J. (Eds.) (2011). *The Duffing Equation: Nonlinear Oscillators and their Behaviour*. Chichester: Wiley. 386 p.
21. Krack M., Gross J. (2019). *Harmonic Balance for Nonlinear Vibration Problems*. Cham: Springer. 159 p. https://doi.org/10.1007/978-3-030-14023-6
22. Krishnapriyan A.S., Gholami A., Zhe S., Kirby R.M., Mahoney M.W. (2021). Characterizing possible failure modes in physics-informed neural networks. *Advances in Neural Information Processing Systems*, 34, 26548-26560. https://arxiv.org/abs/2109.01050
23. Lagaris I.E., Likas A., Fotiadis D.I. (1998). Artificial neural networks for solving ordinary and partial differential equations. *IEEE Transactions on Neural Networks*, 9(5), 987-1000. https://doi.org/10.1109/72.712178
24. Liu D.C., Nocedal J. (1989). On the limited memory BFGS method for large scale optimization. *Mathematical Programming*, 45, 503-528. https://doi.org/10.1007/BF01589116
25. Maclaurin D., Duvenaud D., Adams R.P. (2015). Autograd: effortless gradients in NumPy. In: *ICML 2015 AutoML Workshop*. Lille. https://github.com/HIPS/autograd
26. Nayfeh A.H., Balachandran B. (1995). *Applied Nonlinear Dynamics: Analytical, Computational, and Experimental Methods*. New York: Wiley. 685 p.
27. Nayfeh A.H., Mook D.T. (1979). *Nonlinear Oscillations*. New York: Wiley. 704 p.
28. Raissi M., Perdikaris P., Karniadakis G.E. (2019). Physics-informed neural networks: a deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational Physics*, 378, 686-707. https://doi.org/10.1016/j.jcp.2018.10.045
29. Roache P.J. (2002). Code verification by the method of manufactured solutions. *Journal of Fluids Engineering*, 124(1), 4-10. https://doi.org/10.1115/1.1436090

30. Stoer J., Bulirsch R. (2002). *Introduction to Numerical Analysis*. 3rd ed. New York: Springer. 744 p. https://doi.org/10.1007/978-0-387-21738-3
31. Temesheva S.M. (2020). A modification of algorithms of the Dzhumabaev parametrization method and a numerical method. *International Journal of Information and Communication Technologies*, 2(1), 66-73.
32. Temesheva S.M., Abdimanapova P.B., Borisov D.I. (2021). Ob odnom metode resheniya semeistva nelineinykh kraevykh zadach dlya obyknovennykh differentsial'nykh uravnenii [On a method for solving a family of nonlinear boundary value problems for ordinary differential equations]. *[Bulletin of Abai KazNPU. Series of Physical and Mathematical Sciences]*, 1(73), 70-75. https://doi.org/10.51889/2021-1.1728-7901.09 (In Russ.)
33. Traub J.F., Woźniakowski H. (1979). Convergence and complexity of Newton iteration for operator equations. *Journal of the ACM*, 26(2), 250-258.
34. Ueda Y. (1979). Randomly transitional phenomena in the system governed by Duffing's equation. *Journal of Statistical Physics*, 20(2), 181-196. https://doi.org/10.1007/BF01011513
35. Virtanen P., Gommers R., Oliphant T.E. et al. (2020). SciPy 1.0: fundamental algorithms for scientific computing in Python. *Nature Methods*, 17(3), 261-272. https://doi.org/10.1038/s41592-019-0686-2
36. Wang S., Teng Y., Perdikaris P. (2021). Understanding and mitigating gradient flow pathologies in physics-informed neural networks. *SIAM Journal on Scientific Computing*, 43(5), A3055-A3081. https://doi.org/10.1137/20M1318043
37. Wilson E.B. (1927). Probable inference, the law of succession, and statistical inference. *Journal of the American Statistical Association*, 22(158), 209-212. https://doi.org/10.1080/01621459.1927.10502953